Pergamon

Int. J. Solids Structures Vol. 35, No. 24, pp. 3151–3164, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(98)00008–0

EXACT AND APPROXIMATE SOLUTIONS, AND VARIATIONAL PRINCIPLES FOR STOCHASTIC SHEAR BEAMS UNDER DETERMINISTIC LOADING

N. IMPOLLONIA*
Dipartimento di Costruzioni e Tecnologie Avanzate, Universita' di Messina, Contrada Sperone, Italy 98166

and

I. ELISHAKOFF
Department of Mechanical Engineering, Florida Atlantic University, Boca Raton, Florida 33431-0991, U.S.A.

(*Received* 27 *April* 1997; *in revised form* 10 *December* 1997)

Abstract—Probabilistic response of shear beams with stochastic flexibility, and subjected to deterministic static loads is studied in this paper. The differential equations governing the probabilistic responses, as well as the variational principles for the probabilistic responses are formulated apparently for the first time. New exact solutions are also derived for specific cases. Stochastic versions of Galerkin and Rayleigh–Ritz method are then applied to obtain approximate solutions when exact solution is unfeasible to derive. Both the exact and the approximate solutions possess a unique characteristic: they are applicable to any value of the coefficient of variation. Previous investigations were unable to capture this remarkable characteristic. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Random vibration of shear beams was studied by Masri and Udwadia (1977), Faccioli (1979), Gasparini *et al.* (1981) and Hrwniewicz and Esat (1995). More general case of the Timoshenko beam under random dynamic excitation was studied in recent decade by Elishakoff and Livshits (1989), Singh and Abdelnasser (1992) and Chang (1994). For other relevant works for shear beams one can consult with papers by Tanahashi (1994), Ikeda and Murota (1996) and Won *et al.* (1996).

Inhomogeneity property is ever present in civil engineering problems. Inhomogeneous shear beam has been dealt with by Gazetas (1981) and others. Probabilistic setting was provided by Gasparini *et al.* (1981). Stochasticity of inhomogeneous properties has been introduced, although in a static setting, in the present study. This paper is a companion to two previous studies on exact solutions of beams with stochastic properties (Elishakoff *et al.*, 1995, 1997). We formulate the differential equation that governs the mean displacement, as well as the differential equation obeyed by the covariance function of the displacement response. Then we develop the variational principles for the probabilistic responses, resulting in those differential equations. Some closed form solutions are derived. For the cases when the exact solution is presently unobtainable, the stochastic version of the Galerkin method is applied to the differential equations for the mean displacement and the covariance function. In addition, the stochastic version of the Rayleigh–Ritz method is formulated in conjunction with the uncovered variational principles.

The remarkable feature of the suggested derivation lies in the fact that the present methods are applicable not only to small variations of the stochastic inhomogeneities. Contrary to almost all previous studies, the results are valid for any coefficient of variation.

* Author to whom correspondence should be addressed. E-mail: nicola@maestrale.unime.it.

2. EXACT SOLUTION

The behavior of the shear beam with spatially stochastic shear flexibility

$$f(x) = \frac{k}{GA} \tag{1}$$

subject to deterministic load $q(x)$ is governed by the following differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{f(x)}\frac{\mathrm{d}w(x)}{\mathrm{d}x}\right] = -q(x) \tag{2}$$

where $w(x)$ = the transverse displacement; $f(x)$ = shear flexibility, assumed to be a spatial random field; $G(x)$ = modulus of elasticity in shear; $A(x)$ = cross-sectional area of the beam; and k = a numerical factor (or shear coefficient). Note that if the cross sectional area and its form are functions of x, then k may change accordingly. Therefore, in general, k can be treated as a deterministic function of x. Upon integration of eqn (2) and multiplication by $f(x)$, we get:

$$\frac{\mathrm{d}w(x)}{\mathrm{d}x} = -V(x)f(x) \tag{3}$$

where

$$V(x) = \int_0^x q(x)\,\mathrm{d}x + V_0 \tag{4}$$

is the shear force, and V_0 is a constant of integration representing the shear force at $x = 0$. Hereinafter we assume that the beam is statically determinate. By taking the mathematical expectation of eqn (3), the governing equation for the mean displacement $\bar{w}(x)$ is obtained

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} = -V(x)\bar{f}(x) \tag{5}$$

where $\bar{f}(x) = E[f(x)]$ is the mean flexibility. Dividing eqn (5) by $\bar{f}(x)$ and then differentiating it, an alternative form of governing equation for the mean displacement $\bar{w}(x)$ is obtained

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\bar{f}(x)}\frac{\mathrm{d}\bar{w}(x)}{\mathrm{d}x}\right] = -q(x) \tag{6}$$

where

$$q(x) = \frac{\mathrm{d}V(x)}{\mathrm{d}x} \tag{7}$$

Note that governing eqns (5) and (6) are identical in their forms to those of the beam with equivalent deterministic shear flexibility $\bar{f}(x)$. Subtracting eqn (5) from eqn (3) and multiplying the resulting equation by analogous expression evaluated at the argument y, we get

$$\frac{\mathrm{d}[w(x)-\bar{w}(x)]}{\mathrm{d}x}\frac{\mathrm{d}[w(y)-\bar{w}(y)]}{\mathrm{d}y} = V(x)V(y)[f(x)-\bar{f}(x)][f(y)-\bar{f}(y)] \tag{8}$$

Taking the expectation leads to:

$$\frac{\partial^2 C_w(x,y)}{\partial x\,\partial y} = V(x)V(y)C_f(x,y) \tag{9}$$

where

$$C_w(x,y) = E\{[w(x)-\bar{w}(x)][w(y)-\bar{w}(y)]\} \tag{10}$$

and

$$C_f(x,y) = E\{[f(x)-\bar{f}(x)][f(y)-\bar{f}(y)]\} \tag{11}$$

are the covariance functions of displacements and flexibility, respectively. Dividing eqn (9) by $C_f(x,y)$ and partially differentiating the result with respect to x and y, an alternative form of the governing equation for covariance function $C_w(x,y)$ is obtained

$$\frac{\partial^2}{\partial x\,\partial y}\left[\frac{1}{C_f(x,y)}\frac{\partial^2 C_w(x,y)}{\partial x\,\partial y}\right] = q(x)q(y) \tag{12}$$

Let us consider a shear beam clamped at $x = 0$ and free at $x = L$. The boundary conditions for the mean displacement $\bar{w}(x)$ read

$$\bar{w}(0) = 0; \tag{13a}$$

$$\frac{1}{\bar{f}(L)}\frac{\mathrm{d}\bar{w}(L)}{\mathrm{d}x} = -Q_L \tag{13b}$$

where $Q_L = V(L)$ is the value of the concentrated vertical force at $x = L$. The boundary conditions for the covariance are at $x = 0$ and $x = L$ read, respectively,

$$C_w(0,y) = E\{[w(0)-\bar{w}(0)][w(y)-\bar{w}(y)]\} = 0 \tag{14a}$$

$$\frac{\partial^2 C_w(L,y)}{\partial x\,\partial y} = Q_L V(y)C_f(L,y) \tag{14b}$$

Similarly at $y = 0$ and $y = L$ the boundary conditions read

$$C_w(x,0) = 0 \tag{15a}$$

$$\frac{\partial^2 C_w(x,L)}{\partial x\,\partial y} = Q_L V(x)C_f(x,L) \tag{15b}$$

Solution of eqn (9) is composed of a complementary solution $\psi(x,y)$ and a particular solution $\phi(x,y)$. The complementary solution can be written as follows:

$$\psi(x,y) = h(x)+g(y) \tag{16}$$

where $h(x)$ and $g(y)$ are arbitrary functions of their respective arguments. Boundary conditions in explicit form become

$$\begin{aligned} C_w(x,0) &= h(x)+g(0)+\phi(x,0) = 0 \\ C_w(0,y) &= h(0)+g(y)+\phi(0,y) = 0 \end{aligned} \tag{17}$$

Hence,

$$h(x) = -\phi(x,0)-g(0)$$
$$g(y) = -\phi(0,y)-h(0) \tag{18}$$

Substituting $x = 0$ into eqn (18) we obtain

$$h(0)+g(0) = -\phi(0,0) \tag{19}$$

Thus we arrive at the following expression for the covariance function

$$C_w(x,y) = \phi(x,y)-\phi(x,0)-\phi(0,y)+\phi(0,0) \tag{20}$$

It is remarkable that this general expression depends upon particular solution $\phi(x,y)$ only.

2.1. *Applications*

Consider some examples. Let us specify the loading and stochastic flexibility as follows

$$q(x) = q_0\,; \quad \bar{f}(x) = f_0\,; \quad C_f(x,y) = \bar{f}(x)\bar{f}(y)\left(1-\frac{|x-y|}{L}\right) \tag{21}$$

The shear force has an expression

$$V(x) = q_0(x-L) \tag{22}$$

Introducing non-dimensional coordinates

$$\xi = \frac{x}{L}; \quad \eta = \frac{y}{L} \tag{23}$$

we rewrite the mean displacement as follows

$$\bar{w}(\xi) = q_0 f_0 L^2 \xi\left(1-\frac{\xi}{2}\right) \tag{24}$$

Substituting eqn (22) in eqn (9), the particular solution $\phi(x,y)$ can be obtained by splitting the integration domain into two parts: one in which $x \geqslant y$ and the other with $x \leqslant y$:

$$\phi(x,y) = \int_0^y \int_0^v V(u)V(v)C_f(u,v)\,\mathrm{d}v\,\mathrm{d}u + \int_0^y \int_v^x V(u)V(v)C_f(u,v)\,\mathrm{d}v\,\mathrm{d}u\,; \quad \text{for } x \geqslant y \tag{25}$$

The particular solution for $x \leqslant y$ can be obtained by formal replacement of x by y and y by x, owing to symmetry in x and y. For this case the covariance function coincides with the particular solution in eqn (25) and reads, for $\xi \geqslant \eta$

$$C_w(\xi,\eta) = q_0^2 f_0^2 L^4 \eta\left(\xi-\xi^2+\frac{\xi^3}{3}+\frac{\xi^2\eta}{4}-\frac{\xi^3\eta}{6}-\frac{\eta^2}{3}-\frac{\xi\eta^2}{3}+\frac{\xi^2\eta^2}{6}+\frac{\eta^3}{3}-\frac{\eta^4}{15}\right) \tag{26}$$

For a triangularly distributed load

$$q(x)=\frac{q_1}{L}x \tag{27}$$

we get

$$\bar{w}(\xi)=\frac{q_1}{2}f_0L^2\xi\left(1-\frac{\xi^2}{3}\right) \tag{28}$$

$$C_w(\xi,\eta)=\frac{q_1^2}{4}f_0^2L^4\eta\left(\xi-\frac{\xi^2}{2}-\frac{\xi^3}{3}+\frac{\xi^4}{4}+\frac{\xi\eta}{2}-\frac{\xi^3\eta}{6}-\frac{\eta^2}{3}-\frac{\xi\eta^2}{3}\right.$$
$$\left.+\frac{\xi^2\eta^2}{6}+\frac{\xi^3\eta^2}{9}-\frac{\xi^4\eta^2}{12}-\frac{\xi\eta^3}{4}+\frac{\xi^3\eta^3}{12}+\frac{7\eta^4}{30}-\frac{\eta^6}{42}\right) \tag{29}$$

In the cases when governing equation for the mean displacement $\bar{w}(x)$ and the covariance function $C_w(x,y)$ cannot lend themselves to the exact solution, the approximate methods must be applied. The application of the Galerkin method is demonstrated as follows.

3. STOCHASTIC VERSION OF THE GALERKIN METHOD

For problems incapable of exact solutions we approximate the beam's mean displacement by the expression:

$$\bar{w}(x)=\phi_0(x)+\mathbf{A}^T\mathbf{\Phi}(x) \tag{30}$$

Likewise, we approximate the true covariance function by the following expression

$$C_w(x,y)=\phi_0^*(x,y)+\mathbf{B}^T\mathbf{\Psi}(x)\otimes\mathbf{\Psi}(y) \tag{31}$$

In eqns (30) and (31) **A** and **B** are constant vectors to be determined; $\phi_0(x)$ and $\phi_0^*(x,y)$ are particular functions satisfying non-homogeneous boundary conditions (13a) and (14a)–(15a) for $\bar{w}(x)$ and $C_w(x,y)$, respectively; the components $\phi_j(x)$ of the vector $\mathbf{\Phi}(x)$ are comparison functions satisfying homogeneous boundary condition (13a) for the mean displacement; the components of the vector $\mathbf{\Psi}(x)\otimes\mathbf{\Psi}(y)$ are comparison functions satisfying homogeneous boundary conditions (14a) and (15a) for the covariance of the displacement; the symbol $\otimes$ is the Kronecker product. If $\mathbf{a}=\{a_i\}$ and $\mathbf{b}=\{b_i\}$ are vectors with p and q components, respectively, $\mathbf{a}\otimes\mathbf{b}$ is a vector with $p\times q$ components:

$$(\mathbf{a}\otimes\mathbf{b})^T=[a_1\mathbf{b}^T\ a_2\mathbf{b}^T\ \cdots\ a_{p-1}\mathbf{b}^T\ a_p\mathbf{b}^T]$$
$$=[a_1b_1\ a_1b_2\ \cdots\ a_1b_q\ a_2b_1\ a_2b_2\ \cdots\ a_2b_q\ \cdots\ a_pb_1\ a_pb_2\ \cdots\ a_pb_{q-1}\ a_pb_q] \tag{32}$$

If we choose N comparison functions $\phi_j(x)$ and M functions $\psi_j(x)$, eqns (30)–(31) can be rewritten as follows:

$$\bar{w}(x)=\phi_0(x)+\sum_{j=1}^{N}A_j\phi_j(x) \tag{33}$$

$$C_w(x,y)=\phi_0^*(x,y)+\sum_{j=1}^{M^*}B_j\phi_j^*(x,y) \tag{34}$$

where $M^*=M^2$, $\phi_i^*(x,y)$ are the components of the vector

$$\mathbf{\Phi}^*(x,y) = \mathbf{\Psi}(x) \otimes \mathbf{\Psi}(y) \tag{35}$$

and A_j, B_j are the components of the vectors **A** and **B**. Weighted residual formulas for the mean displacement $\bar{w}(x)$ governed by eqn (5) and for the covariance function $C_w(x,y)$ governed by eqn (9) read

$$\int_0^L \left[\frac{\mathrm{d}\bar{w}(x)}{\mathrm{d}x} + \bar{f}(x)V(x)\right]\phi_i(x)\,\mathrm{d}x = 0 \quad (i = 1,2,\ldots,N) \tag{36}$$

$$\int_0^L \int_0^L \left[\frac{\partial^2 C_w(x,y)}{\partial x\,\partial y} - V(x)V(y)C_f(x,y)\right]\phi_i^*(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0 \quad (i = 1,2,\ldots,M^*) \tag{37}$$

Substitution of eqns (33) and (34) in eqns (36) and (37) yields

$$\sum_{j=1}^{N} K_{ij}^{(1)} A_j = F_i^{(1)} \quad (i = 1,2,\ldots,N) \tag{38}$$

$$\sum_{j=1}^{M^*} S_{ij}^{(1)} B_j = G_i^{(1)} \quad (i = 1,2,\ldots,M^*) \tag{39}$$

where:

$$K_{ij}^{(1)} = \int_0^L \phi_j'(x)\phi_i(x)\,\mathrm{d}x \tag{40}$$

$$F_i^{(1)} = -\int_0^L \phi_0'(x)\phi_i(x)\,\mathrm{d}x - \int_0^L f(x)V(x)\phi_i(x)\,\mathrm{d}x \tag{41}$$

$$S_{ij}^{(1)} = \int_0^L \int_0^L \phi_{j,xy}^*(x,y)\phi_i^*(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{42}$$

$$G_i^{(1)} = \int_0^L \int_0^L \phi_{0,xy}^*(x,y)\phi_i^*(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_0^L \int_0^L C_f(x,y)V(x)V(y)\phi_i^*(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{43}$$

Once the sets of eqns (38) and (39) are solved, the coefficient A_j and B_j, respectively, denoted as $A_j^{(1)}$ and $B_j^{(1)}$, must be substituted into eqns (33) and (34) to arrive at the final expressions of the mean displacement and the covariance function, respectively.

Alternatively, the Galerkin method can be formulated using eqns (6) and (12), instead of eqns (5) and (9). In this case the functions $\phi_0(x)$ and $\phi_0^*(x,y)$, $\phi_j(x)$ and $\phi_j^*(x,y)$ are required to satisfy all boundary conditions, non-homogeneous and homogeneous ones, respectively. Weighted residual formulas read

$$\int_0^L \left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\bar{f}(x)}\frac{\mathrm{d}\bar{w}(x)}{\mathrm{d}x}\right] + q(x)\right\}\phi_i(x)\,\mathrm{d}x = 0 \quad (i = 1,2,\ldots,N) \tag{44}$$

$$\int_0^L \int_0^L \left\{\frac{\partial^2}{\partial x\,\partial y}\left[\frac{1}{C_f(x,y)}\frac{\partial^2 C_w(x,y)}{\partial x\,\partial y}\right] - q(x)q(y)\right\}\phi_i^*(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0 \quad (i = 1,2,\ldots,M^*) \tag{45}$$

Substitution of eqns (33) and (34) into eqns (44) and (45) yields two sets of linear algebraic equations formally similar to eqns (38) and (39), provided that the expressions in the equations are now supplemented by the superscript (2), as opposed to the superscript (1) in the former formulation

$$K_{ij}^{(2)} = \int_0^L \phi_i(x) \frac{d}{dx}\left[\frac{1}{\bar{f}(x)}\phi_j'(x)\right] dx \tag{46}$$

$$F_i^{(2)} = -\int_0^L \phi_i(x) \frac{d}{dx}\left[\frac{1}{\bar{f}(x)}\phi_0'(x)\right] dx - \int_0^L q(x)\phi_i(x)\,dx \tag{47}$$

$$S_{ij}^{(2)} = \int_0^L \int_0^L \phi_i^*(x,y) \frac{\partial^2}{\partial x\,\partial y}\left[\frac{1}{C_f(x,y)}\phi_{j,xy}^*(x,y)\right] dx\,dy \tag{48}$$

$$G_i^{(2)} = -\int_0^L \int_0^L \phi_i(x,y) \frac{\partial^2}{\partial x\,\partial y}\left|\frac{1}{C_f(x,y)}\phi_{0,xy}^*(x,y)\right| dx\,dy + \int_0^L \int_0^L q(x)q(y)\phi_i^*(x,y)\,dx\,dy \tag{49}$$

The question arises whether the use of the identical comparison functions to the two different, alternative forms of the different equations for the specified probabilistic characteristics will yield the same answer. Specifically, the question is if the use of identical comparison functions to eqns (5) and (6) for the mean displacement, or to eqns (9) and (12) for the covariance will result in the same result. The reply to this question is negative. This implies that in general case $A_j^{(1)} \neq A_j^{(2)}$ and $B_j^{(1)} \neq B_j^{(2)}$. However, one can show that utilization of the Galerkin method to eqns (6) and (12), and of the Petrov–Galerkin method to eqns (5) and (9), can lend to coinciding results. Indeed, using approximation (33) for $\bar{w}(x)$ and (34) for $C_w(x,y)$ for evaluating the residuals but the different orthogonalizing functions, namely, $\phi_{i,x}(x)/f(x)$ and $\phi_{i,xy}^*(x,y)/C_f(x,y)$ we obtain the following weight residual formulas for the Petrov–Galerkin method

$$\int_0^L \left[\frac{d\bar{w}(x)}{dx} + \bar{f}(x)V(x)\right]\frac{\phi_{i,x}(x)}{f(x)} dx = 0 \quad (i = 1,2,\ldots,N) \tag{50}$$

$$\int_0^L \int_0^L \left[\frac{\partial^2 C_w(x,y)}{\partial x\,\partial y} - V(x)V(y)C_f(x,y)\right]\frac{\phi_{i,xy}^*(x,y)}{C_f(x,y)} dx\,dy = 0 \quad (i = 1,2,\ldots,M^*) \tag{51}$$

where $\phi_{i,x} = \partial\phi_i/\partial x$, $\phi_{i,xy}^*\partial^2\phi = \partial^2\phi_i^*/\partial x\,\partial y$. The proof is straightforward and is obtained through integration by part of the left side of eqns (44) and (45).

3.1. *Applications*

Let us consider cases of either constant or triangularly distributed loads applied to the shear beam with stochastic flexibility specified in eqn (21). We use eqns (33) and (34) with a single term approximation, $N = M = 1$, and solve eqns (38) and (39). As comparison function the exact displacement of the corresponding deterministic beam is selected. Since the differential equation for the mean displacement coincides in its form with its counterpart for the deterministic beam it is of no surprise that we hit the exact solution.

Let us now discuss the evaluation of the displacement covariance function. When the load is constant we have

$$\phi_0(\xi) = \phi_0^*(\xi, \eta) = 0; \quad \phi_1(\xi) = \psi_1(\xi) = q_0 f_0 L^2 \xi \left(1 - \frac{\xi}{2}\right) \tag{52}$$

and

$$C_w(\xi, \eta) = \tfrac{47}{252} q_0^2 f_0^2 L^4 \xi\eta(-2+\xi)(-2+\eta) \tag{53}$$

For a triangular load

$$\phi_0(\xi) = \phi_0^*(\xi, \eta) = 0; \quad \phi_1(\xi) = \psi_1(\xi) = \frac{q_1}{2} f_0 L^2 \xi \left(1 - \frac{\xi^2}{3}\right) \tag{54}$$

with attendant covariance function reading

$$C_w(\xi, \eta) = \tfrac{107}{5148} q_1^2 f_0^2 L^4 \xi\eta(-3+\xi^2)(-3+\eta^2) \tag{55}$$

For comparison between exact covariance functions $C_w(\xi, \eta)$ derived in Section 2 and their approximations given by eqns (53) and (55) we have chosen to compute the variances portrayed in Figs 1 and 2. As is seen, a single-term approximation turns out to be excellent.

When the displacement of the corresponding deterministic solution is not available or is cumbersome, trigonometric functions can be chosen as comparison functions. To verify the convergence of the method we have studied the stepped beam with cross-sectional area $A/2$ for $0 \leqslant x < L/2$, and equal A for $L/2 \leqslant x \leqslant L$. The probability characteristics of flexibility are given by

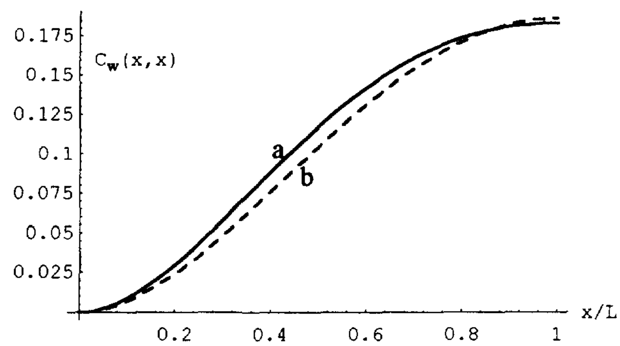

Fig. 1. Displacement variance for cantilever beam subjected to uniformly distributed load, normalized by $q_0^2 f_0^2 L^4$: (a) exact solution; (b) Galerkin method.

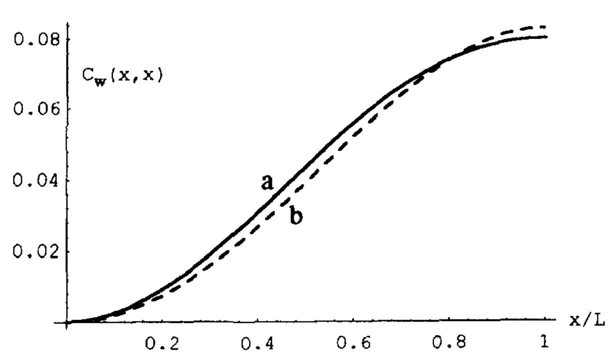

Fig. 2. Displacement variance for cantilever beam subjected to triangularly distributed load, normalized by $q_1^2 f_0^2 L^4$: (a) exact solution; (b) Galerkin method.

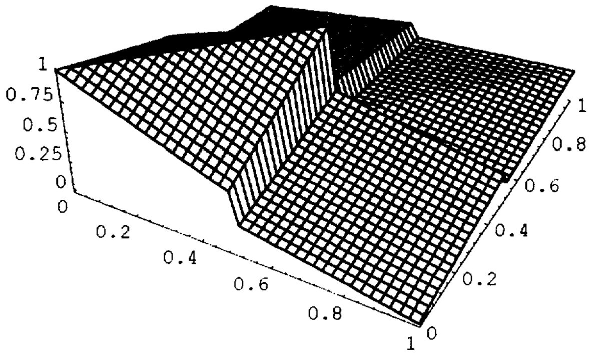

Fig. 3. Covariance function of the flexibility for the stepped beam.

$$\bar{f}(x) = f_0 \quad \text{for } 0 \leqslant x < L/2$$
$$\bar{f}(x) = f_0/2 \quad \text{for } L/2 \leqslant x \leqslant L$$
$$C_f(x,y) = \bar{f}(x)\bar{f}(y)\left(1 - \frac{|x-y|}{L}\right) \tag{56}$$

The load is fixed at q_0. The covariance of flexibility is depicted in Fig. 3. The generic comparison function reads

$$\phi_i(\xi) = \psi_i(\xi) = q_0 f_0 L^2 \sin\left[\frac{(2i-1)\pi\xi}{2}\right]; \quad (i = 1, \ldots, N = M) \tag{57}$$

while

$$\phi_0(\xi) = \phi_0^*(\xi, \eta) = 0 \tag{58}$$

Solving eqns (38) and (39), for $N = M = 1$ results in

$$\bar{w}(\xi) = 0.3563 q_0 f_0 L^2 \sin\left(\frac{\pi\xi}{2}\right) \tag{59}$$

$$C_w(\xi,\eta) = 0.0970 q_0^2 f_0^2 L^4 \sin\left(\frac{\pi\xi}{2}\right)\sin\left(\frac{\pi\eta}{2}\right) \tag{60}$$

whereas for $N = M = 2$ the results read

$$\bar{w}(\xi) = q_0 f_0 L^2 \left[0.4735 \sin\left(\frac{\pi\xi}{2}\right) + 0.0391 \sin\left(\frac{3\pi\xi}{2}\right)\right] \tag{61}$$

$$C_w(\xi,\eta) = q_0^2 f_0^2 L^4 \left\{0.1915 \sin\left(\frac{\pi\xi}{2}\right)\sin\left(\frac{\pi\eta}{2}\right) + 0.0197\left[\sin\left(\frac{3\pi\xi}{2}\right)\sin\left(\frac{\pi\eta}{2}\right) + \sin\left(\frac{\pi\xi}{2}\right)\sin\left(\frac{3\pi\eta}{2}\right)\right] + 0.0026 \sin\left(\frac{3\pi\xi}{2}\right)\sin\left(\frac{3\pi\eta}{2}\right)\right\} \tag{62}$$

For $N = M = 3$ the approximate responses read

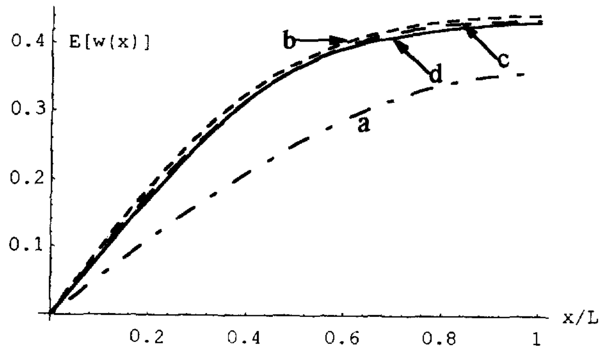

Fig. 4. Mean displacement for stepped beam, subjected to uniformly distributed load, obtained by the Galerkin method, normalized by $q_0 f_0 L^2$: (a) $N = 1$; (b) $N = 2$; (c) $N = 3$; (d) $N = 4$; $N = 5$ and $N = 6$.

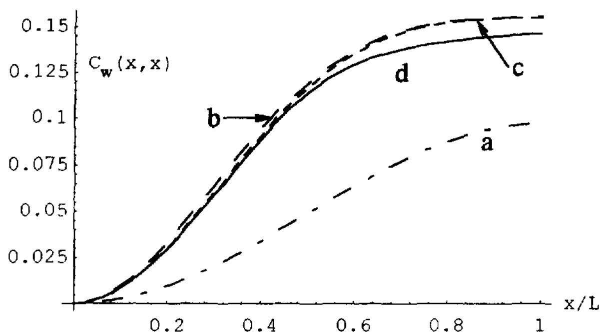

Fig. 5. Displacement variance for stepped beam, subjected to uniformly distributed load, obtained by the Galerkin method, normalized by $q_0^2 f_0^2 L^4$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$; $M = 5$ and $M = 6$.

$$\bar{w}(\xi) = q_0 f_0 L^2 \left[0.4828 \sin\left(\frac{\pi\xi}{2}\right) + 0.0437 \sin\left(\frac{3\pi\xi}{2}\right) + 0.0027 \sin\left(\frac{5\pi\xi}{2}\right)\right] \tag{63}$$

$$\begin{aligned} C_w(\xi,\eta) = q_0^2 f_0^2 L^4 \Bigg\{ & 0.1938 \sin\left(\frac{\pi\xi}{2}\right)\sin\left(\frac{\pi\eta}{2}\right) + 0.0213\left[\sin\left(\frac{3\pi\xi}{2}\right)\sin\left(\frac{\pi\eta}{2}\right)\right. \\ & \left. + \sin\left(\frac{3\pi\eta}{2}\right)\sin\left(\frac{\pi\xi}{2}\right)\right] + 0.0006\left[\sin\left(\frac{5\pi\xi}{2}\right)\sin\left(\frac{\pi\eta}{2}\right) + \sin\left(\frac{5\pi\eta}{2}\right)\sin\left(\frac{\pi\xi}{2}\right)\right] \\ & + 0.0037 \sin\left(\frac{3\pi\xi}{2}\right)\sin\left(\frac{3\pi\eta}{2}\right) + 0.0004\left[\sin\left(\frac{5\pi\xi}{2}\right)\sin\left(\frac{3\pi\eta}{2}\right)\right. \\ & \left. + \sin\left(\frac{5\pi\eta}{2}\right)\sin\left(\frac{3\pi\xi}{2}\right)\right] + 0.0001 \left(\frac{5\pi\xi}{2}\right)\sin\left(\frac{5\pi\eta}{2}\right)\Bigg\} \end{aligned} \tag{64}$$

Results were also calculated for larger value of N and M. The obtained mean and variance functions for the displacement are plotted in Figs 4 and 5 for different values of N and M. We can clearly see that values of N and M equal or larger than three result in virtually identical curves.

4. STOCHASTIC VARIATIONAL PRINCIPLES

The variational principle for the mean displacement $\bar{w}(x)$ corresponding to eqn (6) consists in the requirement that the following functional takes a stationary value:

$$\Pi_1 = \int_0^L \frac{1}{2\bar{f}(x)}\left(\frac{d\bar{w}(x)}{dx}\right)^2 dx - \int_0^L q(x)\bar{w}(x)\,dx + Q_L\bar{w}(L) \tag{65}$$

The variational principle for the covariance function $C_w(x,y)$ corresponding to eqn (12) consists in the requirement the following functional takes a stationary value:

$$\Pi_2 = \int_0^L\int_0^L \frac{1}{2C_f(x,y)}\left(\frac{\partial^2 C_w(x,y)}{\partial x\,\partial y}\right)^2 dx\,dy - \int_0^L\int_0^L q(x)q(y)C_w(x,y)\,dx\,dy + \int_0^L Q_L C_w(L,y)q(y)\,dy + \int_0^L Q_L C_w(x,L)q(x)\,dx - Q_L^2 C_w(L,L) \tag{66}$$

One can check by using the Lagrange–Euler equations that eqns (6) and (12) follow from eqns (65) and (66), respectively. Variational principles allow one to develop an associated Rayleigh–Ritz method, and the finite element technique. In the former the trial functions are required to satisfy only the kinematic boundary conditions. Hence, the mean and covariance function of the displacement can be expressed as follows

$$\bar{w}(x) = \sum_{j=1}^{N} A_j\chi_j(x) \tag{67}$$

$$C_w(x,y) = \sum_{j=1}^{M^*} B_j\chi_j^*(x,y) \tag{68}$$

The functions $\chi_j^*(x,y)$ are the components of the vector

$$X^*(x,y) = \mathbf{\Omega}(x) \otimes \mathbf{\Omega}(y) \tag{69}$$

$\chi_j(x)$, and the component $\omega_j(x)$ of the vector $\mathbf{\Omega}(x)$ are trial functions corresponding to the mean displacement. The unknowns A_j and B_j can be obtained solving the following systems

$$\sum_{j=1}^{N} K_{ij}^{(3)}A_j = F_i^{(3)} \quad (i = 1,2,\ldots,N) \tag{70}$$

$$\sum_{j=1}^{M^*} S_{ij}^{(3)}B_j = G_i^{(3)} \quad (i = 1,2,\ldots,M^*) \tag{71}$$

where following notations have been utilized

$$K_{ij}^{(3)} = \int_0^L \frac{1}{\bar{f}(x)}\chi_i'(x)\chi_j'(x)\,dx \tag{72}$$

$$F_i^{(3)} = -\int_0^L q(x)\chi_i(x)\,dx - Q_L\chi_i(L) \tag{73}$$

$$S_{ij}^{(3)} = \int_0^L\int_0^L \frac{1}{C_f(x,y)}\chi_{i,xy}^*(x,y)\chi_{j,xy}^*(x,y)\,dx\,dy \tag{74}$$

$$G_i^{(3)} = \int_0^L \int_0^L q(x)q(y)\chi_i^*(x,y)\,\mathrm{d}x\,\mathrm{d}y - \int_0^L \chi_i^*(L,y)q(y)\,\mathrm{d}yQ_L - \int_0^L \chi_i^*(x,L)q(x)\,\mathrm{d}xQ_L + Q_L^2\chi_i^*(L,L) \quad (75)$$

4.1. *Applications*

Let us again consider a clamped-free beam with stochastic flexibility given in eqn (21). As trial functions the following functions are chosen:

$$\chi_i(\xi) = \omega_i(\xi) = \xi^i \quad (76)$$

When the load is constant, fixing N and M at two we hit the exact mean displacement, given in eqn (28). For the covariance function we get

$$C_w(\xi,\eta) = q_0^2 f_0^2 L^4[0.8024\xi\eta - 0.4349(\xi^2\eta + \xi\eta^2) + 0.2474\xi^2\eta^2] \quad (77)$$

Equation (77) turns out to be a good approximation of the exact solution given in eqn (26). For a triangular load, using $N = M = 2$, we get

$$\bar{w}(\xi) = q_1 f_0 L^2 \xi(0.5833 - 0.25\xi) \quad (78)$$

$$C_w(\xi,\eta) = q_1^2 f_0^2 L^4[0.2953\xi\eta - 0.1564(\xi^2\eta + \xi\eta^2) + 0.0939\xi^2\eta^2] \quad (79)$$

In this case the two term approximation for the mean displacement is very accurate. Yet, the accuracy for the covariance function is insufficient. Therefore for its reliable estimation three term approximation is needed. For $M = 3$ the approximate covariance function read

$$C_w(\xi,\eta) = q_1^2 f_0^2 L^4[0.3342\xi\eta - 0.31214(\xi^2\eta + \xi\eta^2) + 0.0959(\xi^3\eta + \xi\eta^3) + 0.7067\xi^2\eta^2 - 0.3816(\xi^3\eta^2 + \xi^2\eta^3) + 0.2343\xi^3\eta^3] \quad (80)$$

Comparison between exact variance functions and those obtained by eqn (77) and eqns (79) and (80) is performed in Figs 6 and 7. As is seen the three term approximation turns out to be a very good one. To make a comparison between the numerical results obtained, when the load is constant, by the Galerkin and the Rayleigh–Ritz method, the exact variance function and those obtained by eqn (53) and (77) are plotted in Fig. 8.

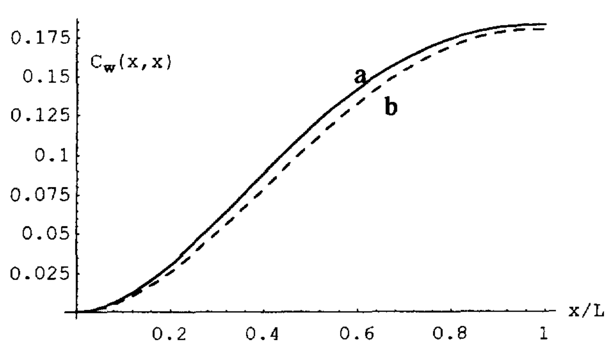

Fig. 6. Displacement variance for cantilever beam subjected to uniformly distributed load, normalized by $q_0^2 f_0^2 L^4$: (a) exact solution; (b) Rayleigh–Ritz method.

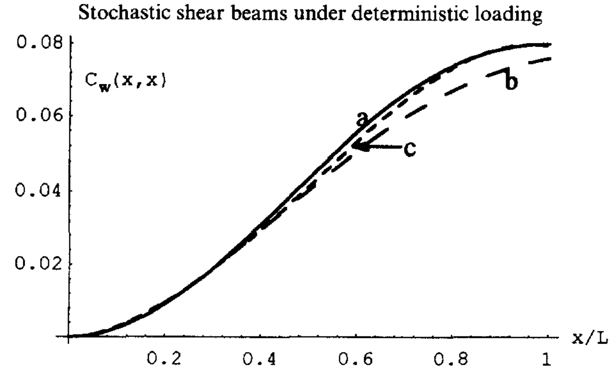

Fig. 7. Displacement variance for cantilever beam subjected to triangularly distributed load, normalized by $q_1^2 f_0^2 L^4$: (a) exact solution; (b) Rayleigh–Ritz method, $M = 2$; (c) Rayleigh–Ritz method, $M = 3$.

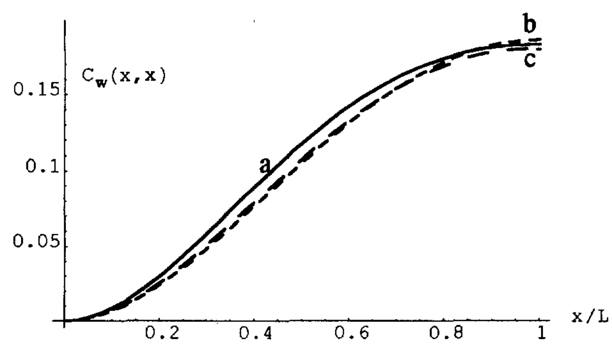

Fig. 8. Displacement variance for cantilever beam subjected to uniformly distributed load, normalized by $q_1^2 f_0^2 L^4$: (a) exact solution; (b) Rayleigh–Ritz method; (c) Galerkin method.

5. CONCLUSION

The first and second order probabilistic characteristics of the deflection fields of statically determinate shear beam with randomly varying shear stiffness subject to deterministic loadings are analytically determined. For problems that appear to be presently incapable of exact solution, the classical method of Galerkin has been suitably generalized to deal with stochastic shear beams. Moreover, two novel variational principles and the associated Rayleigh–Ritz method have been derived for determination of probabilistic characteristics of the response. The proposed methods are valid for any value of variation of the stochastic parameters.

The works generalizing present technique for both random flexibility and random loading, as well the stochastic version of the finite element method are underway and will be reported elsewhere.

Acknowledgements—I. Elishakoff acknowledges support by the U.S. Army Research Office (Program Director, Dr G. Anderson). The study was conducted when N. Impollonia was a Visiting Research Scholar at the Department of Mechanical Engineering of the Florida Atlantic University under the auspices of the University of Messina, Italy. This support is gratefully appreciated. Any opinions, findings and recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the sponsors.

REFERENCES

Chang, T.-P. (1994) Deterministic and random vibration of an axially loaded Timoshenko beam resting on an elastic foundation, *Journal of Sound and Vibration* **178**, 55–66.

Elishakoff, I. and Livshits, D. (1989) Some closed form solutions in random vibrations of Timoshenko beams. *Probabilistic Engineering Mechanics* **4**, 49–54.

Elishakoff, I., Impollonia, N. and Ren, Y. J. (1997) New exact solutions for randomly loaded beams with stochastic flexibility. *International Journal of Solids and Structures*, submitted.

Elishakoff, I., Ren, Y. J. and Shinozuka, M. (1995) Some exact solutions for the bending of beams with spatially stochastic stiffness. *International Journal of Solids and Structures* **32**, 2315–2327.

Faccioli, E. (1979) A stochastic approach to soil amplification. *Bull. Seism. Soc. America* **66**, 1277–1291.

Gasparini, D. A., Debchandbhury, A. and Gazetas, G. (1981) Random vibration of cantilever shear beam. *Earthquake Engineering and Structural Dynamics* **9**, 599–612.

Gazetas, G. (1981) A new dynamic model for earth dams evaluated through case histories. *Soils and Foundations* (*Japan Society of Soil Mechanics and Foundation Engineering*) **21**, 67–78.

Hryniewicz, Z. and Esat, I. E. (1995) Insight to cutoff frequency of shear beam on random foundation. *ASME Structural Dynamics and Vibration*, *Petroleum Division*, pp. 87–95. ASME Press, New York.

Ikeda, K. and Murota, K. (1996) Bifurcation as sources of uncertainty in soil shearing behavior. *Soils and Foundations* **36**, 73–84.

Masri, S. F. and Udwadia, F. (1977) Transient response of a shear beam to correlated random boundary excitation. *Journal of Applied Mechanics* **44**, 487–491.

Tanahashi, H. (1994) Probability-based prediction of differential settlements of structures using Timoshenko beam on Pasternack model. *Soils and Foundations* **34**, 77–89.

Singh, M. P. and Abdelnasser, A. S. (1992) Random response of symmetric cross-ply composite beams with arbitrary boundary conditions. *AIAA Journal* **30**, 1081–1088.

Wolfram, S. (1996) *Mathematica, A System for Doing Mathematics by Computer*. Cambridge University Press.

Won, A. Y. J., Pires, J. A. and Haroun, M. A. (1996) Stochastic seismic performances evaluation of tuned liquid column dampers. *Earthquake Engineering and Structural Dynamics* **25**, 1259–1274.